Subject: Development of Fukui Equation **Date:** August 28, 2012
Reference(s): **Author:** Jeff Crawford

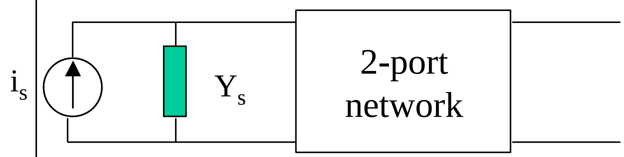

Figure 1

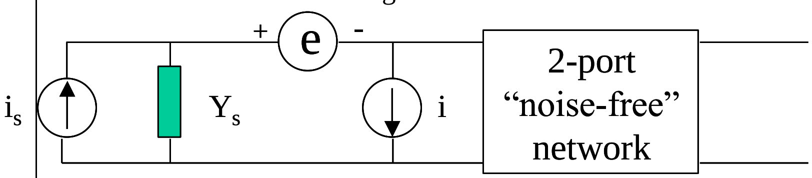

Figure 2

Figure 1 below represents a typical two-port network with any inherent noise mechanisms represented in Figure 2 where “e” and “i” represent uncorrelated noise sources and the network is considered noise free.

The noise figure for Figure 2 is the ratio of the total output noise power per unit BW available at the output to that portion of the total output noise power generated by the input termination.

The total output noise power is proportional to $\overline{i_{sc}^2}$ while the noise power due to the source alone is proportional to $\overline{i_s^2}$. (i_{sc} = short-circuit current at the input terminals of the noise free network).

Shorting the terminals just before the “noise-free” two-port we are able to calculate i_{sc}:

$$
\begin{aligned}
i_s &= eY_s + i + i_{sc} \\
i_{sc} &= i_s - \left(i + eY_s\right)
\end{aligned}
\qquad (1,2)
$$

$$\overline{i_{sc}^2} = \overline{\left[i_s - \left(i + eY_s\right)\right]\left[i_s - \left(i + eY_s\right)\right]^*}$$

Now there is zero correlation between the noise source and the input generator so $e i_s^* = e^* i_s \equiv 0$. Taking this into account gives:

$$\overline{i_{sc}^2} = \overline{\left[i_s - \left(i + eY_s\right)\right]\left[i_s - \left(i + eY_s\right)\right]^*} = \overline{i_s i_s^* - i_s^*\left(i + eY_s\right) - i_s\left(i + eY_s\right)^* + \left|i + eY_s\right|^2}$$

$$\overline{i_{sc}^2} = \overline{\left|i_s\right|^2} + \overline{\left|i + eY_s\right|^2}$$

The noise figure then becomes equal to: $F = \dfrac{\overline{\left|i_{sc}\right|^2}}{\overline{\left|i_s\right|^2}} = 1 + \dfrac{\overline{\left|i + eY_s\right|^2}}{\overline{\left|i_s\right|^2}}$

Normally there is some degree of correlation between noise sources “e” and “i”. We can divide “i” up into one part, i_u, not correlated with “e” and a second part, $i - i_u$, which is correlated to “e”.

$i - i_u$ is related to “e” by a correlation admittance, Y_c.

$$i = i_u + i_c \qquad i = i_u + Y_c e \qquad [6]$$

where $Y_c = G_c + j\,B_c$

Also $\overline{e\,i^*} = \overline{e\left(i_u + Y_c\,e\right)^*} = \overline{e\,i_u} + Y_c^*\overline{|e|^2} = Y_c^*\overline{|e|^2}$ [7] because e and i_u are uncorrelated.

We can write the following noise powers:

$$\overline{e^2} = 4kT_oR_nB$$
$$\overline{i_u^2} = 4kT_oG_uB \qquad \text{where } R_n \text{ is the equivalent noise resistance of "e"}$$
$$\overline{i_s^2} = 4kT_oG_sB$$

Returning now to [5] we can carry the expression for F a little farther:

$$F = 1 + \frac{\overline{|i + eY_s|^2}}{\overline{i_s^2}} = 1 + \frac{\overline{(i + eY_s)(i + eY_s)^*}}{\overline{i_s^2}}$$
$$F = \frac{\overline{|i|^2 + ei^*Y_s + e^*iY_s^* + |Y_s|^2}\,\overline{e^2}}{\overline{i_s^2}} + 1 \qquad [11]$$

Looking at [11] term by term:

$$\overline{|i|^2} = \overline{(i_u + eY_c)(i_u + eY_c)^*} = \overline{|i_u|^2} + \overline{Y_c e i_u^*} + \overline{Y_c^* e^* i_u} + |Y_c|^2\,\overline{e^2}$$
$$= \overline{|i_u|^2} + |Y_c|^2\,\overline{e^2} \qquad [12]$$
$$\overline{|i|^2} = 4kT_oB\left(G_u + |Y_c|^2 R_n\right)$$

$$\overline{ei^*Y_s} = \overline{e(i_u + eY_c)^* Y_s} = \overline{ei_uY_s} + \overline{|e|^2 Y_c^*Y_s} = \overline{e^2}Y_sY_c^*$$
$$\overline{e^*iY_s^*} = \overline{e^*(i_u + eY_c)Y_s^*} = \overline{e^*i_uY_s^*} + \overline{e^*eY_cY_s^*} = \overline{e^2}Y_cY_s^* \qquad [13],[14]$$

Combining [12]-[14] and substituting we obtain

$$F = \frac{4kT_oB\left(G_u + |Y_c|^2 R_n + Y_c^*Y_sR_n + Y_cY_s^*R_n + |Y_s|^2 R_n\right) + 1}{4kT_oBG_s} = 1 + \frac{G_u}{G_s} + \frac{R_n}{G_s}|Y_s + Y_c|^2 \qquad [15]$$
$$F = 1 + \frac{G_u}{G_s} + \frac{R_n}{G_s}\left((G_s + G_c)^2 + (B_s + B_c)^2 \right)$$

We see that the overall noise figure is very dependent upon the admittance of the driving source.

To develop the Fukui equation from [15] we must solve for the minimum attainable noise figure, i.e. where does

$$\frac{\partial F}{\partial G_s} = 0 \qquad [16]$$

One criterion to minimize F is to set $(B_s + B_c)^2 = 0$ or $B_c = -B_s$. [17]

For convenience we define the optimum source admittance to be $Y_m = G_m + jB_m$ [18]

Then $B_m = -B_c$.

To evaluate G_m:

$$\frac{\partial F}{\partial G_s}\bigg|_{B_s=B_m} = \frac{\partial}{\partial G_s}\left\{\frac{G_u + R_n\left(G_s+G_c\right)^2}{G_s}\right\} = 0$$

$$R_n G_s^2 - R_n G_c^2 - G_u = 0 \qquad [20]$$

or

$$G_s = G_m = \left[G_c^2 + \frac{G_u}{R_n}\right]^{1/2}$$

Therefore, the optimum noise figure, to be called F_o, occurs at the optimum source admittance Y_m.

$$Y_m = G_m + jB_m \ \ where \ \ G_m = \left[G_c^2 + \frac{G_u}{R_n}\right]^{1/2}$$

$$B_m = -B_c$$

Substituting $G_m + jB_m$ into [15] gives the minimum noise figure possible:

$$F_o = 1 + 2R_n\left[G_c + \left(G_c^2 + \frac{G_u}{R_n}\right)^{1/2}\right] \qquad [21]$$

The last step to arrive at Fukui's equation is to substitute F_o into [15] and eliminate all direct dependence upon the correlation admittances G_c and Y_c.

Beginning with [15]:

$$F = 1 + \frac{G_u}{G_s} + \frac{R_n}{G_s}\left\{\left(G_s+G_c\right)^2 + \left(B_s+B_c\right)^2\right\}$$

$$= 1 + \frac{G_u}{G_s} + \frac{R_n}{G_s}\left(G_s^2 + 2G_sG_c + G_c^2 + B_s^2 + 2B_sB_c + B_c^2\right)$$

$$= 1 + \frac{R_n}{G_s}\left(G_s^2 + 2G_sG_c + \left(G_c^2 + \frac{G_u}{R_n}\right) + B_s^2 + 2B_sB_c + B_c^2\right)$$

$$G_m^2 \qquad\qquad set\ B_m = -B_c$$

From [21] we solve for G_c:

$$G_c = \frac{F_o - 1}{2R_n} - G_m \qquad [22]$$

$$F = 1 + \frac{R_n}{G_s}\left[G_s^2 + 2G_s\left(\frac{F_o-1}{2R_n} - G_m\right) + G_m^2 + \left(B_s - B_m\right)^2\right]$$

$$= 1 + \frac{R_n}{G_s}\left(\frac{F_o-1}{2R_n}\right)2G_s + \frac{R_n}{G_s}\left[G_s^2 - 2G_sG_m + G_m^2 + \left(B_s-B_m\right)^2\right]$$

$$F = F_o + \frac{R_n}{G_s}\left[\left(G_s-G_m\right)^2 + \left(B_s-B_m\right)^2\right]$$

$$F = F_o + \frac{R_n}{G_s}\left[\left(G_s - G_m\right)^2 + \left(B_s - B_m\right)^2\right]$$
$$F = F_o + \frac{R_n}{G_s}\left|Y_s - Y_m\right|^2$$

F_o = absolute minimum attainable noise figure
Y_m = optimal noise match
R_n = equivalent noise resistance

Knowing the four noise parameters F_o, R_n, G_m, B_m completely characterizes the possible performance.