A Circuit & Systems Technique:
The Hilbert Transform

by *Jeff Crawford*

The Hilbert transform can be a powerful tool[1] in the hands of any circuit designer or systems analyst. In work related to networks it is possible only from resistance vs. frequency information to determine the associated reactance for a minimum-phase realization. Equally powerful in systems analysis is the ability to determine the phase component vs. frequency for a transfer function based solely on gain magnitude versus frequency data. With the phase information a qualitative assessment of the inherent group delay (with the assumption of a minimum-phase network) is at one's fingertips through the use of a central difference approximation to the first derivative, i.e.

$$\tau_{gd} = -\frac{d\,\phi(\,\omega\,)}{d\,\omega} \qquad [\,1\,]$$

The piecewise-continuous Hilbert transform will be used to synthesize the imaginary part of an impedance from only the real part, and the respective phase from magnitude-only data versus frequency for a system transfer function. This can be extended to determine the associated group delay using [1] above.

The Piecewise Hilbert Transform

Each resistance and reactance value of a network can be represented by a finite series of the form given in [2] and [3].

$$R_Q(\,w\,) = \sum_{k=0}^{N} a_k(\,\omega\,)\,r_k \qquad r_0 = R_Q(\,0\,)$$

$$X_Q(\,w\,) = \sum_{k=1}^{N} b_k\,(\,w\,)\,r_k \qquad [2,\,3]$$

Each r_k is a positive or negative real number describing the known resistance excursion of the k^{th} straight line-segment between the frequency breakpoints ω_{k-1} and ω_k of the piecewise linear representation of R(ω) by R_Q(ω). Referencing Figure 1, each r_k is determined by evaluating the difference between the resistance values R_k and R_{k-1} at frequencies ω_k and ω_{k-1}. In the case of a system transfer function, each r_k is the difference between neighboring values of the transfer function's magnitude, | H(jω_k) | and | H(jω_{k-1}) |.

Referencing equation [2], it is not necessary to calculate the set { a_k } to determine the reactance

In the case of determining the reactance from the resistance as a function of frequency, the { r_k } and { a_k } are readily determined because all the needed

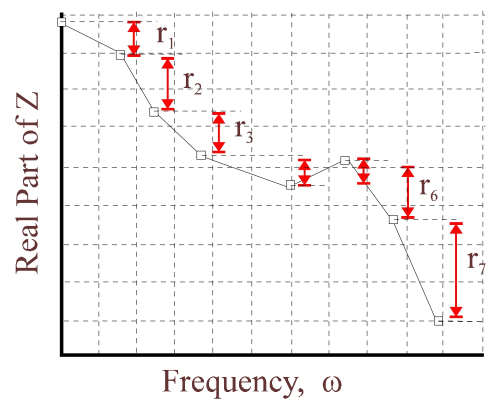

Figure 1

Piecewise Representation of Real(Z)

versus frequency function X_Q(ω). The { a_k } take on differing values depending upon what frequency is of interest. For frequencies below ω_{k-1}, ω_k is identically zero, while for frequencies above ω_k, the a_k is identically 1.

$$a_k = \begin{cases} 0 & \omega_k < \omega_{k-1} \\ \dfrac{\omega - \omega_{k-1}}{\omega_k - \omega_{k-1}} & \omega_{k-1} < \omega < \omega_k \\ 1 & \omega \geq \omega_k \end{cases} \qquad [\,4\,]$$

for k = 1 to n

information is provided. It is necessary, however, to use the Hilbert transform relation in [5] to determine the associated reactive component, X_Q(ω), at each respective frequency.

$$X_Q(\omega) = \frac{1}{\pi}\int_0^{\infty} \frac{dR}{d\omega}\ln\left[\frac{y+\omega}{y-\omega}\right] dy \quad [\,5\,]$$

An expression for R is readily available from [1], therefore the operation indicated in [5] above can be carried out explicitly. We start by rewriting [2] as

$$R_Q(\omega) = r_0 + \sum_{k=1}^{n} a_k(\omega) r_k$$

Using the expression in [1] developed for R_Q, the derivative operation is applied and the following simplifications identified.

$$\frac{d\;R_Q(\omega)}{d\,\omega} = \sum_{k=1}^{n} \frac{\partial\{a_k(\omega)\}}{\partial\omega} r_k$$

$$[6,7,8] = \begin{cases} \displaystyle\sum_{k=1}^{n} r_k \frac{\partial(1)}{\partial\omega} = 0 & If\;\; \omega \ge \omega_k \\ \displaystyle\sum_{k=1}^{n} r_k \frac{\partial}{\partial\omega}\left(\frac{\omega-\omega_{k-1}}{\omega_k-\omega_{k-1}}\right) & \omega_{k-1} < \omega \le \omega_k \\ \displaystyle\sum_{k=1}^{n} r_k \frac{\partial}{\partial\omega}(0) = 0 & \omega < \omega_{k-1} \end{cases}$$

Review of equations [6 - 8] shows that the only element entering into the calculation with a non-zero contribution is that for which the frequency ω lies in between the frequencies ω_{k-1} and ω_k. Performing the derivative operation on equation [7] gives the intermediate result in [9].

$$\frac{d\;R_Q(\omega)}{d\,\omega} = \sum_{k=1}^{n} r_k \frac{\partial}{\partial\omega}\left(\frac{\omega-\omega_{k-1}}{\omega_k-\omega_{k-1}}\right) = \sum_{k=1}^{n} r_k \left(\frac{1}{\omega_k-\omega_{k-1}}\right)$$

[9]

Substituting this expression into the integral relationship of [5] gives the following result.

$$X_Q(\omega) = \frac{1}{\pi}\int_0^{\infty} \left(\sum_{k=1}^{n}\left(\frac{r_k}{\omega_k-\omega_{k-1}}\right)\right) \ln\left(\left|\frac{y+\omega}{y-\omega}\right|\right) dy$$

[10]

The linear operations of summation and integration may be interchanged to give the complete expression required to evaluate the imaginary part from the real part.

$$X_Q(\omega) = \sum_{k=1}^{n} r_k \left[\frac{1/\pi}{\omega_k-\omega_{k-1}} \int_{\omega_{k-1}}^{\omega_k} \ln\left(\left|\frac{y+\omega}{y-\omega}\right|\right)\right] dy \qquad [\,11\,]$$

The part of the expression in brackets can be viewed to be equal to the coefficients { b_k } with the final simplification for X_Q being expression [12].

$$X_Q(\omega) = \sum_{k=1}^{n} b_k(\omega)\, r_k \qquad [\,12\,]$$

The integral in [11] has a closed-form solution[2], given in [13] and [14], which further simplifies the end result. Furthermore, a minor transformation of variables given in [15], can be employed[3] which prevents errors when the argument of the log function is one or zero.

$$b_k(\omega) = \frac{1/\pi}{\omega_k - \omega_{k-1}} \left[(y+\omega)\ln(y+\omega) - (y-\omega)\ln(y-\omega) \right]_{y=\omega_{k-1}}^{y=\omega_k} \qquad [\ 13\ ]$$

or more concisely:

$$b_k(\omega) = \frac{B(\omega,\omega_k) - B(\omega,\omega_{k-1})}{\pi(\omega_k - \omega_{k-1})} \qquad [\ 14\ ]$$

$$B(\omega,\overline{\omega}) = \overline{\omega}\left[(X+1)\ln(X+1) + (X-1)\ln(|X-1|) - 2X\ln(X) \right]$$

$$X \equiv \frac{\omega}{\overline{\omega}} \qquad X \neq 1; \qquad X \neq 0 \qquad B(\omega,\overline{\omega})\Big|_{\overline{\omega}=0} \equiv 0 \qquad [\ 15\ ]$$

Actual Applications of the Hilbert Transform

In the opening paragraph of this article mention was made of the utility of the Hilbert technique in several areas. As an example and an aide to give increased insight into the technique, two different applications of the method will be outlined here. In the first method the real part of a rational polynomial in s (s = jω) will be used to generate the minimum phase realization reactance, and compared with the exact reactance calculated from the seed polynomial used. In the second application the magnitude of a rational polynomial in s will be used to illustrate the utility of the technique to derive the phase of a transfer function based solely on the magnitude versus frequency data. First principles are used to design a fifth-order chebyshev bandpass filter from which its magnitude response is obtained. The synthesized phase is then used to determine the filter's group delay. These calculations are then compared to an "exact" method of group delay calculation for this family of filters.

Case 1: *Synthesis of Minimum Phase Reactance from Resistance versus Frequency Data*

The expression used in this example is [16]. The coefficients were selected to display a rapid change in both the real and imaginary parts of the polynomial.

$$H(\omega) = \frac{s+2.5}{s^2 - 0.76\,s + 5} \qquad [\ 16\ ]$$

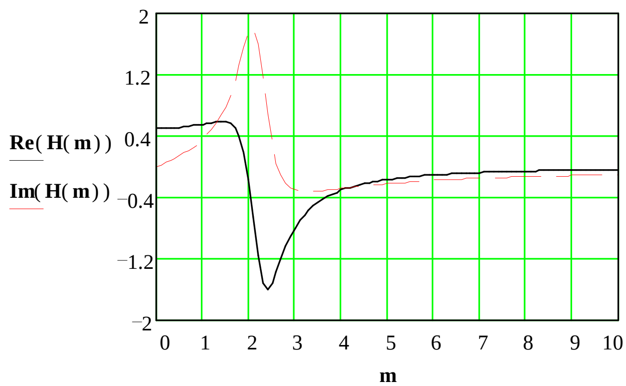

Figure 2
Impedance versus Frequency of Equation [16].

Performing the Hilbert transform operation, using as input the real part of [16] over frequency, gives the results shown in Figure 3. In this figure, the exact calculation for the reactance is compared with that derived from the Hilbert technique. Figure 4 illustrates the comparison of phase angle for the impedance, using as comparison the exact calculations from the beginning relationship in [16].

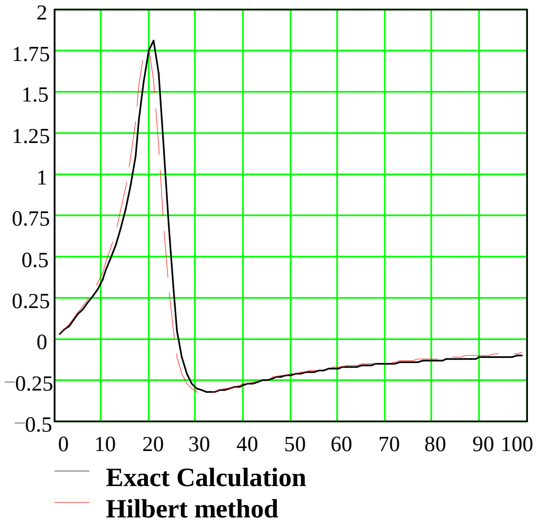

Figure 3
Synthesis of Reactive Component Comparison

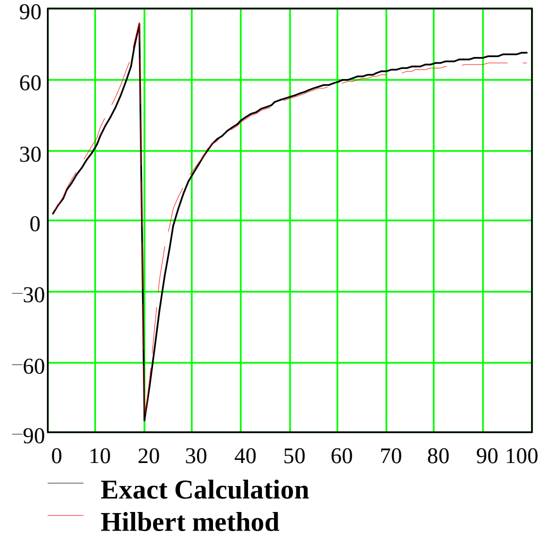

Figure 4
Phase Angles of Impedances Compared

Case 2: *Synthesis of Phase Angle from Magnitude-Only Data versus Frequency*

In this example the magnitude versus frequency performance of a fifth-order chebyshev bandpass filter is used to synthesize the respective phase over frequency. Although the details will not be discussed, a model for the bandpass filter which includes Q-losses is used to give a more accurate magnitude representation of the true bandpass filter response; the required calculations are illustrated in the Appendix.

The filter is described as fifth-order, passband ripple of 0.5 dB, unloaded resonator Q of 500, and ripple corner frequencies of 45.0 and 55.0 MHz, respectively. The magnitude response is illustrated in Figure 5 over frequency.

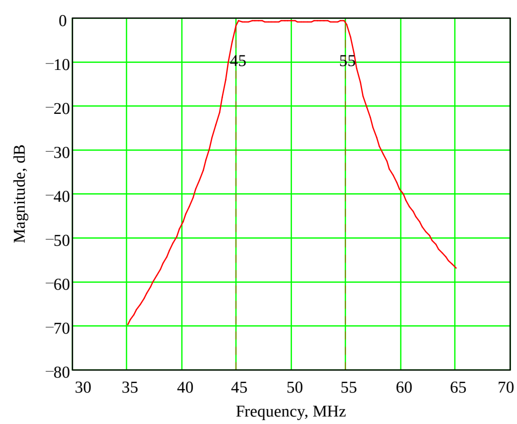

Figure 5
Bandpass Response of Candidate Chebyshev Filter

Performing the Hilbert technique on the magnitude response illustrated in Figure 5 gives the respective phase shown in Figure 6. As seen in this figure, the phase is ≈ *linear*, corrupted through the passband by the large passband ripple used in the filter's design. Had the ripple used in the design been more reasonable to present a better in-band VSWR, i.e. ripple on the order of < 0.1 dB, the phase response would be much more monotonic without variation. A comparison of the group delay calculated with the Hilbert technique, and the group delay of the filter using the known pole locations of the chebyshev filter, is shown in Figure 7. The gross structure of the two curves is very similar, differing primarily in the finer details.

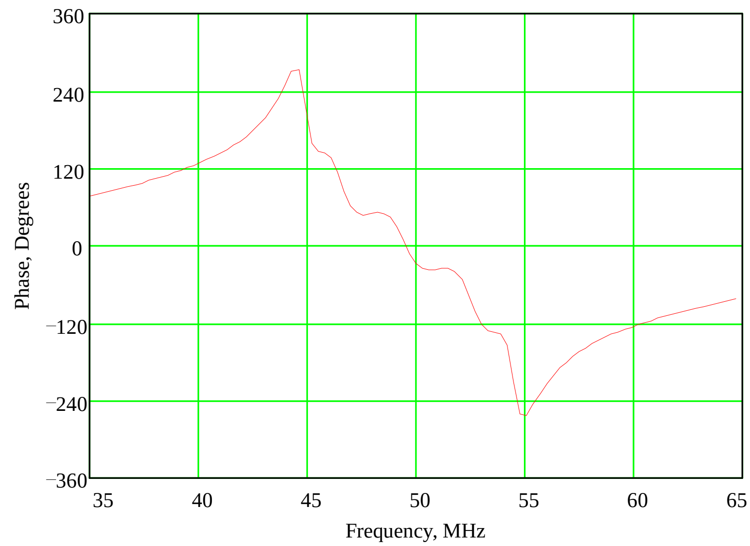

Figure 6
Phase Response Through Filter Passband

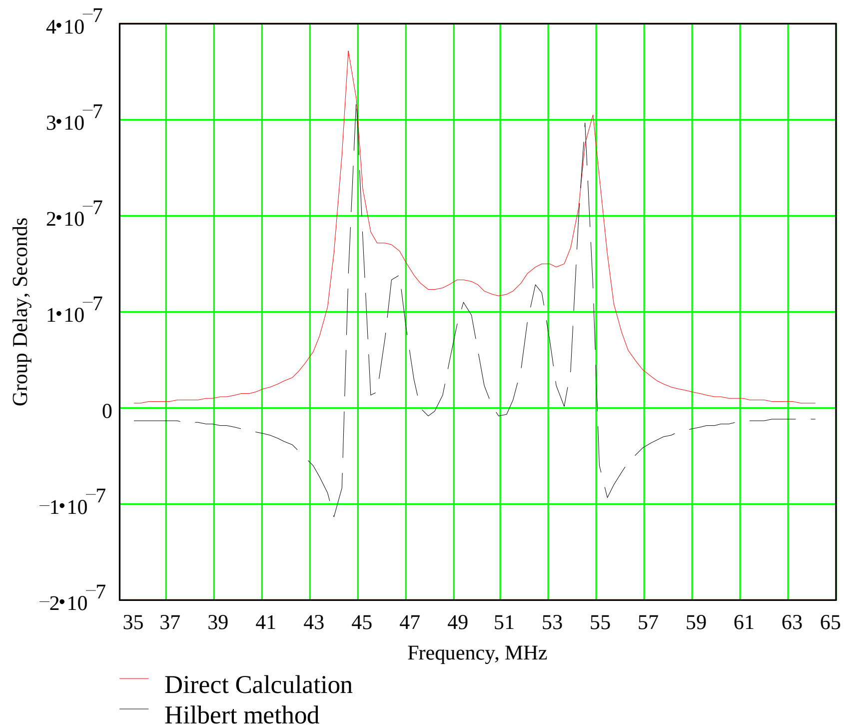

Figure 7
Comparison of Actual Group Delay and Group Delay Calculated via the Hilbert Transform

A central differences approximation to the first derivative, equation [17], is used to calculate the group delay from the phase information.

$$\frac{d\ f(x_0)}{dx} = \frac{-f(x_2) + 8\ f(x_1) - 8\ f(x_{-1}) + f(x_{-2})}{12(\Delta x)} \qquad [17]$$

[1]"A New Approach to Gain-Bandwidth Problems," Carlin, Herbert J., IEEE Circuits and Systems, April, 1977, pp. 170 - 175.
[2] Network Analysis and Feedback Amplifier Design, Bode, W.H., Van Nostrand, 1945, pg. 319.
[3] Circuit Design Using Personal Computers, Cuthbert, T.R., John Wiley, 1984, pp. 219-221.