Noise Characterization Methods Made Intuitive
by Jeff Crawford

Introduction

Most RF designers are cognizant of the fundamental noise figure relationship[1] for cascaded two-port networks given in [1], and perhaps the relationship for the noise performance of a single two-port[1,7,8] given in [2]. The first expression calculates the overall cascaded noise figure for a matched system wherein the input and output of each elementary stage are identically equal to 50 Ω. The second relationship identifies departures from optimum noise figure performance for a two-port as a function of source termination admittance. What deviations from these exact theories occur in a less than ideal cascaded system, and how do modern-day simulators determine degraded noise figure performance? How are noise figure calculations affected if incomplete noise parameters are input to a simulator such as Omnisys or Libra? The answers to these questions, supported by an intuitive approach to the accompanying theoretical background, are presented in the pages which follow.

$$F = F_1 + \frac{F_2 - 1}{G_1} + \frac{F_3 - 1}{G_1 G_2} + \frac{F_4 - 1}{G_1 G_2 G_3} + \ldots \qquad [1]$$

$$NF = NF_{\min} + \frac{r_n}{G_s} \left| Y_s - Y_o \right|^2 \qquad [2]$$

Brute-Force Approach to Noise Calculations

It is instructive to approach the entire subject of noise calculations from fundamental principles, gaining momentum as increasingly powerful circuit concepts and theories are applied. Our study of noise relationships begins with the introduction of the concept of available power gain as contrasted with insertion gain, transducer gain, or other related definitions of gain. Available power gain is valuable[2,3] due to its inherent property of readily allowing the cascading of multiple networks. Transducer gain and insertion gain, for example, do not in general have this cascading property. In the case of N networks in cascade, the available power gain of the k^{th} network is measured with the output resistance of the (k-1)th network as the source impedance. In a completely matched system where source and load impedances are all theoretically equal, i.e. 50 Ω, the various power gain definitions converge to equivalent results. Several common definitions of power gain are given in Table III.

The concept of available power gain may be derived from the simplistic circuit of Figure 1, where the selection of load resistor, R_L, is done in such a manner which facilitates maximum power delivery to the load by the source. With the annotations as illustrated in Figure 1, the power delivered to the load is given in [3]. Taking the partial derivative of [3] with respect to R_L and setting equal to zero shows that making $R_L = R_s$ maximizes the percentage of power delivered to the load. In general terms, maximum power is delivered to the load when $Z_L = Z_S^*$. The absolute power available from the source decreases as the load increases. The absolute power in watts delivered to the load is also at its highest value when $R_s \equiv R_L$. When the normalized load impedance exceeds 1.0, a higher percentage normalized power is delivered to the load, but the absolute power in the load itself is less. Figure 2 illustrates this relationship graphically.

Paralleling this expression for available power in a circuit, Nyquist showed[4] in 1928, that the mean-square noise voltage from a resistor was governed mathematically by [5]. His theory showed further that a noisy resistor could be represented equivalently by an rms noise voltage in series with a noiseless resistor R as shown in Figure 3.

$$\overline{e_n^2} = 4\,k\,T\,R\,df \qquad [5]$$

where

$k \equiv$ Boltzmann's constant $\quad T \equiv$ temperature

$R \equiv$ resistance $\quad df \equiv$ bandwidth, Hz.

Therefore, it is possible to replace each resistor in a network with a noise-free resistance in cascade with a noise voltage source of amplitude

$$\overline{e_n} = \sqrt{4\,k\,T\,R\,df} \qquad [6]$$

Note, that rearranging [5] into [7] shows the thermal noise power, kT df, is the available noise power from the network, Also of importance is that the thermal noise power available from the resistor is independent of the resistance, R, being governed entirely by the temperature T.

$$\frac{\overline{e_n^2}}{4R} = k\,T\,df \qquad [7]$$

As one further example consider Figure 4 in which several different resistors are arranged arbitrarily. We seek the cumulative noise power available at the terminals of this one-port. Each noise voltage source may be treated individually, one at a time, and its respective open-circuit voltage[5] determined at the output terminals of the one port. Due to the non-coherence properties of noise from each resistor, each rms noise voltage may be squared and summed directly to give the total noise power at the one-port's output terminals. This is shown mathematically in [8],

$$\overline{e_T^2} = \sum_k \overline{e_k^2}\,|A_k|^2 \quad [8]$$

where each $|A_k|$ is the transfer function describing the open-circuit voltage appearing at the output terminals for each resistor. The mathematics for Figure 4 as given in equation [8] give the following results:

$$\begin{aligned}\overline{e_T^2} &= 4\,k\,T\;df\left[\frac{R_1\,R_2^2}{(R_1+R_2)^2} + \frac{R_1^2\,R_2}{(R_1+R_2)^2} + R_3\right] \\ \overline{e_T^2} &= 4\,k\,T\;df\left[\frac{R_1\,R_2}{(R_1+R_2)} + R_3\right] = 4\,k\,T\;df\;R_T\end{aligned} \qquad [9]$$

Two important points must be illuminated given the result in [9]. First of all, the application of Thevenin and Norton transformations would give the same resultant R_T as in [9]. But most importantly, the noise power appearing at the output terminals of the one-port, $\frac{\overline{e_T^2}}{4\,R_T}$, does not depend on R_T whatsoever, as we identified earlier in [7]. In other words, regardless of the transformations, i.e. the $|A_k|$s, involved in determining the individual open-circuit noise voltages at the terminals, the available noise power at the output terminals is always kT df; always! This latter fact allows us to perform the final simplification to the noise factor calculation procedure.

By definition the noise figure is the ratio of the total noise power available per unit bandwidth at the output to the total noise power available at the input from the source alone. Writing this mathematically in terms of signal-to-noise ratios at the input and output gives [10].

$$NF = \frac{(S/N)_{in}}{(S/N)_{out}} = \frac{S_{in}}{N_{in}}\,\frac{N_{out}}{S_{out}} = \frac{S_{in}\,N_{out}}{N_{in}\,G_A\,S_{in}} = \frac{N_{out}}{G_A\,N_{in}} \qquad [10]$$

The final simplification for [10] comes by virtue of the fact that the noise power available from the source alone ($4kT_{in}$ df), as well as the noise power available at the network's output ($4kT_{out}$ df), are identical and equal to kT df in passive circuits when T_{in} and T_{out} are identical ($T_{in} \equiv T_{out} \equiv T$). This generalization is not accurate for active circuits where expression [2] applies; more details on this subsequently. Therefore, the noise factor of the network is identically the reciprocal of the available power gain, G_A. In the case of passive circuits of identical temperature, the noise figure in dB is equivalent to the network loss in dB.

$$NF = \frac{1}{G_A} \qquad NF_{dB} = 10\log_{10}\left(1/G_A\right) \qquad [11]$$

If the ambient temperature of the circuit network is T_o, it is possible to determine a second temperature T_e which, by virtue of Nyquist's Theorem expressed in [5], accounts for the additional degradation in noise figure performance by the network. Bear in mind that degraded noise figure does not always imply increased noise power production in a circuit, but instead could indicate less source noise being transfered to the load, while at the same time the network continuing to produce the same kT noise power in the load. The effective temperature may be calculated by a slight modification of [10] wherein T_e accounts for noise inherent in the network itself.

$$NF = \frac{N_{out}}{G_A N_{in}} = \frac{k\left(T_o + T_e \right) G_A \; df}{G_A \, k \, T_o \;\; df} = 1 + \frac{T_e}{T_o} \qquad [12]$$

where T_e represents the thermal noise contribution of the network moved to the input side, and the network then considered noiseless.

Modification of [12] gives an expression for the equivalent noise temperature of the network, T_e, in terms of the noise factor, NF.

$$T_e = T_o \left(NF - 1 \right) \qquad [13]$$

Summarizing, the definition of available power gain is given by [14].

$$G_A = \frac{\text{Power avail from network}}{\text{Power avail from source}} \qquad [14]$$

In the case of the network or the source, the available power can be determined through the use of [4] where V_s is the Thevenin equivalent open-circuit voltage and R_s, the Thevenin equivalent resistance. In those cases where the source and accompanying network may be resolved through multiple Thevenin and Norton transformations into an equivalent source voltage and series resistor, it is possible to determine the available noise power from the network. Forming the ratio of this noise power to the noise power available from the source alone gives the noise factor for the circuit. An example will illustrate the procedure.

Example Noise Factor Calculation

Consider the passive circuit case shown in Figure 5, composed of reactive and resistive components. Through a successive application of Thevenin and Norton transformations this network is simplified to a modified voltage and source resistance, and load. Figure 6 is the resultant simplification composed of the Thevenin equivalent driving the original load resistance, R_L. The expressions in [15] identify the modified source resistance Z_T and voltage source, V_T. The noise factor calculation proceeds in a very straight forward manner as indicated in [16].

$$Z_1(\omega) = \frac{j\omega \; LR}{j\omega L + R} \qquad V_T = V_S \frac{Z_1(\omega) Z_3(\omega)}{R_S \, Z_2(\omega)}$$

$$Z_2(\omega) = Z_1(\omega) + R \qquad [15]$$

$$Z_T(\omega) = \frac{Z_2(\omega)}{1 + j\omega C \, Z_2(\omega)}$$

$$NF = \frac{\text{Noise pwr avail network}}{\text{Noise pwr avail source}} = \frac{\dfrac{V_T^2}{4\,\mathrm{Re}\left(Z_T \right)}}{\dfrac{V_S^2}{4\,R_S}} = \frac{\dfrac{V_S^2 \left| \dfrac{Z_1(\omega) Z_3(\omega)}{R_S \, Z_2(\omega)} \right|^2}{4\,\mathrm{Re}\left(Z_T \right)}}{\dfrac{V_S^2}{4\,R_S}} = \frac{\left| \dfrac{Z_1(\omega) Z_3(\omega)}{R_S \, Z_2(\omega)} \right|^2 R_S}{\mathrm{Re}\left(Z_T(\omega) \right)} = G_A \qquad [16]$$

Use of [16] and the accompanying element values of Figure 7 give a circuit noise figure versus frequency characteristic as shown in Figure 8. The dotted line is the asymptotic noise figure approached at infinite frequency. This noise figure is the same as for a series resistor and is given by [17]. To facilitate your following through with you own calculations, the following discrete points are offered:

$$NF_{dB} = 10\log_{10}\left(1 + \frac{R}{R_s}\right) \qquad [17]$$

Enhancements to the Noise Figure Calculation

As intuitive as the above calculation is to the reader, the technique is predisposed to many unattractive features for increasingly complex circuits. In order to streamline the technique, as well as serve as a precursor for yet to be discussed noise figure calculation methods, the use of ABCD matrices is in order. Through various well-known parameter transformations[6,7,8], S-parameters or Y-parameters could be used with identical results. The ABCD matrix technique as employed here is ideal for simple cascaded elements.

Table II depicts several typical lumped elements and their respective ABCD matrix representation[9]. For analysis purposes of the circuit in Figure 7, only the matrices for series R, and parallel L and C are required. The composite ABCD matrix for the cascade of these three elements is given in [18]. The voltage gain for this cascaded system is given in [19], expressed using elements of the composite matrix A. The available power gain is then given in [20]. The noise figure is then given using G_P in place of G_A in expression [11]. The interested reader can perform these calculations and verify that both techniques give exactly identical numerical results.

$$A = \begin{bmatrix} 1 & 0 \\ \dfrac{-j}{\omega L} & 1 \end{bmatrix} \begin{bmatrix} 1 & R \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ j\omega C & 1 \end{bmatrix} \qquad [18]$$

$$G_V = \frac{V_{out}}{V_{in}} = \frac{A_{1,1}\,R_L + A_{1,2} + A_{2,1}\,R_S R_L + A_{2,2}\,R_S}{R_L} \qquad [19]$$

$$G_P = 10\log_{10}\left(4\frac{R_S}{R_L}\left|G_V\right|^2\right) \qquad [20]$$

Next Generation Noise Analysis Techniques

Thus far noise contributions of each element have been handled individually, requiring considerable work to evaluate the cumulative effect. A more powerful method which segregates the problem into noise and noise-free representations provides the vehicle needed for larger, more complex analyses. Figure 9 is the “black-box” two-port representation for a linear network in which all noise sources are combined indistinguishably with other network attributes. Figure 10 illustrates pictorially the partitioning of the network into its constituent parts[10], noise and noise-free. This repackaging of the problem is possible using a generalization of Thevenin’s theorem. Since a noise-free network connected to a terminal pair does not change the signal-to-noise ratio the noise factor of the overall-all network is equal to that of the noise network alone.

Other representations leading to a different separation of the internal sources from the twoport are possible[11]. The representation in Figure 11 is particularly useful in that four noise parameters are easily derived from single frequency measurements of the twoport noise factor as a function of input match. Of further benefit is the fact that neither gain nor input conductance enter into the noise factor expression in terms of the four noise parameters. The presence of both noise sources at the input is in particular advantageous when calculating the noise figures of amplifiers. The two-port representation in Figure 11 is used to derive, from first principles, the fundamental noise figure relationship expressed earlier in [2].

$$[A] = k\,T_{phys}\Big[\,[\,I\,] - [S][S^*]^T\,\Big]$$

$$\begin{bmatrix} A_{1,1} & A_{1,2} \\ A_{2,1} & A_{2,2} \end{bmatrix} = k\,T_{phys}\left[\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} S_{1,1} & S_{1,2} \\ S_{2,1} & S_{2,2} \end{bmatrix}\begin{bmatrix} S_{1,1}{}^* & S_{1,2}{}^* \\ S_{2,1}{}^* & S_{2,2}{}^* \end{bmatrix}^T\right]$$

$$N_{1,1} = k\,T_o\left(\frac{4\,R_n}{\left|1+G_{opt}\right|^2} - F_{\min} + 1\right)$$

$$N_{1,2} = k\,T_o\,\frac{4\,R_n\left|G_{opt}\right|}{\left|1+G_{opt}\right|^2}\big[\cos(\,\phi\,) + j\sin(\,\phi\,)\big]$$

$$\omega here \quad \phi = \pi - \angle G_{opt}$$

$$N_{2,1} = k\,T_o\,\frac{4\,R_n\left|G_{opt}\right|}{\left|1+G_{opt}\right|^2}\big[\cos(\,\phi\,) - j\sin(\,\phi\,)\big]$$

$$\omega here \quad \phi = \pi - \angle G_{opt}$$

$$N_{2,2} = k\,T_o\left\{F_{\min} - 1 + \frac{4\,R_n\left|G_{opt}\right|^2}{\left|1+G_{opt}\right|^2}\right\}$$

$$\omega here \quad \phi = \pi - \angle G_{opt}$$

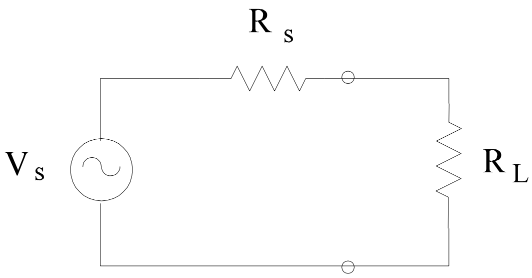


Figure 1

$$P_{Load} = \frac{V_s^2 R_L}{(R_L + R_s)^2} \quad [3]$$

$$\frac{\partial P_{Load}}{\partial R_L} = \frac{2 R_L}{R_L + R_s} - 1$$

Max Power to Load when $R_L \equiv R_s$

$$P_{Avail} = \frac{V_s^2}{4 R_s} \quad [4]$$

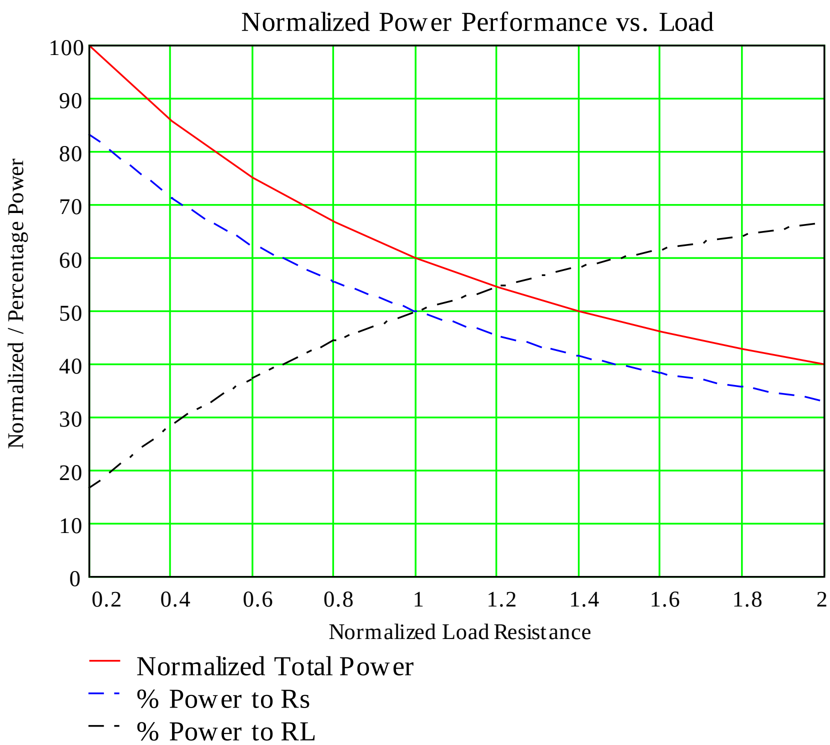


Figure 2

When Source ≡ Load, Equal Power is in Load and Source

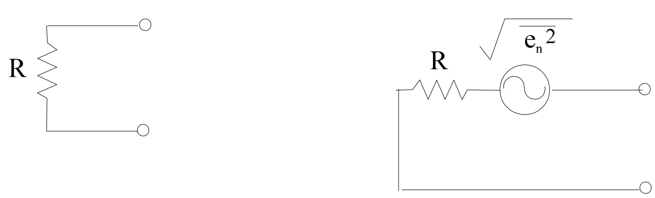
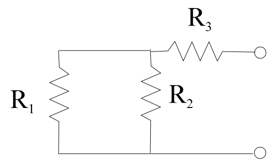


Figure 3

Typical Resistor

Noise-free Resistor and Accompanying Thermal Noise Voltage Source



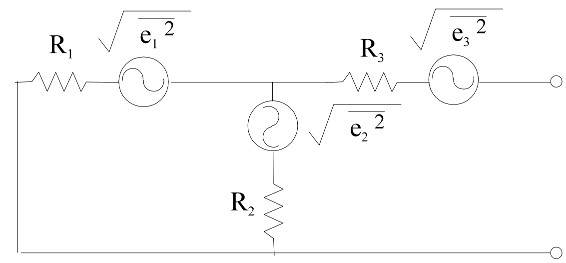


Figure 4

Typical Resistor Network

Noise-free Resistors and Accompanying Thermal Noise Voltage Sources

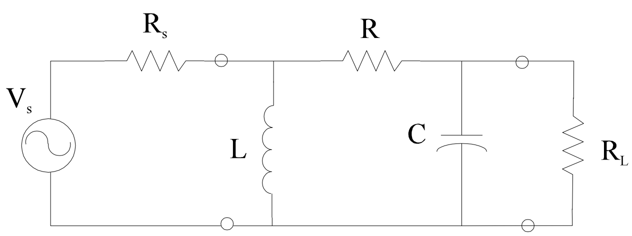


Figure 5
Example Passive Circuit for Noise Calculation

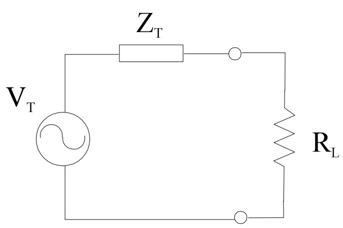


Figure 6
Thevenin Equivalent Circuit

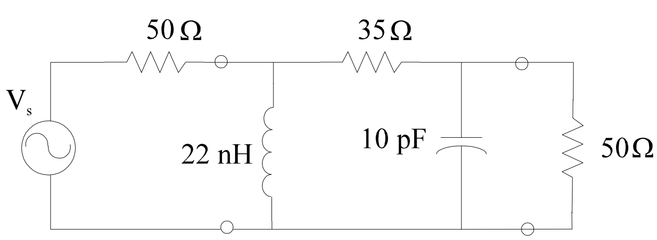


Figure 7
Populated Passive Circuit for Noise Calculation

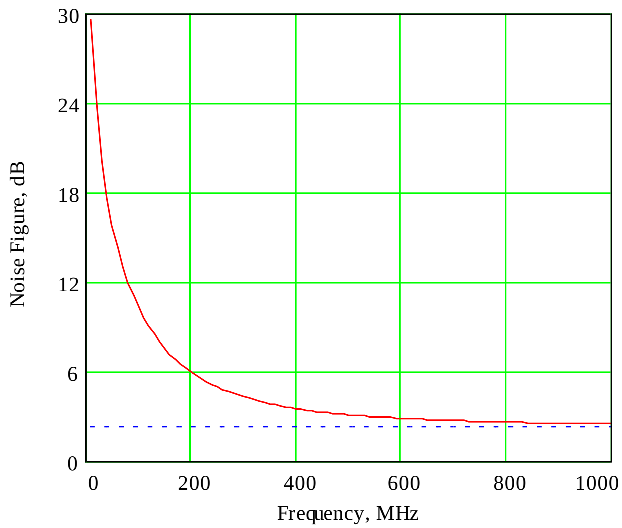


Figure 8
Noise Figure Calculations for Network in Figure 7

Frequency, MHz	NF, dB	Frequency, MHz	NF,dB
1	49.618	200	6.00937
10	29.6264	300	4.3419
50	15.8359	500	3.152
100	10.358	1000	2.5324

Table 1
Sample Numerical Results for Noise Figure of Figure 7

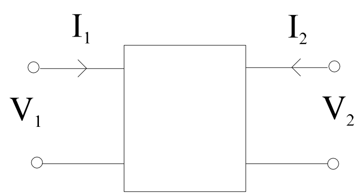


Figure 9
Two-Port Network With All Noise Sources Internal

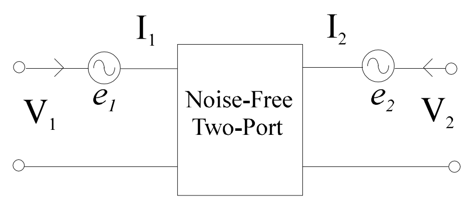


Figure 10
Segregation of Noise and Noise-Free Circuit Elements

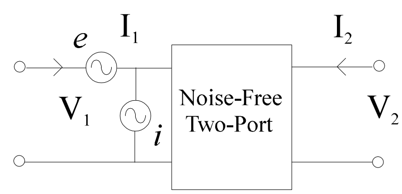


Figure 11
Two-Port Network With Both Noise Sources at the Input

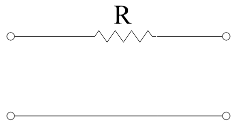
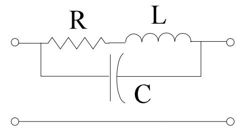
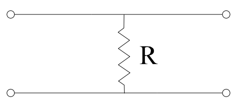
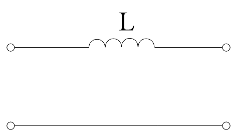
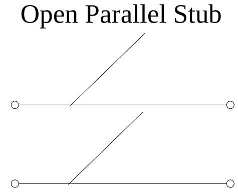
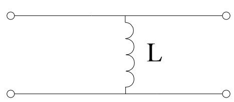
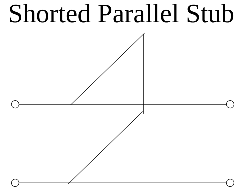
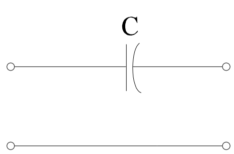
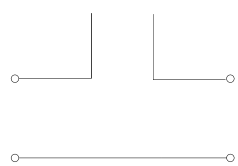
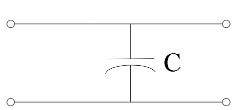
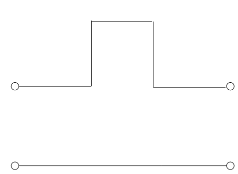
Element Type	ABCD Representation	Element Type	ABCD Representation
Series R 	$\begin{bmatrix} 1 & R \\ 0 & 1 \end{bmatrix}$	Series RL/Parallel C 	$\begin{bmatrix} 1 & R_B + j\,I_B \\ 0 & 1 \end{bmatrix}$ $R_B = \dfrac{R}{\left(1+\omega^2 LC\right)^2 + \left(\omega RC\right)^2}$ $I_B = \dfrac{\omega\left[L\left(1-\omega^2 LC\right) - R^2 C\right] R_B}{R}$
Parallel R 	$\begin{bmatrix} 1 & 0 \\ \dfrac{1}{R} & 1 \end{bmatrix}$	Transmission Line 	$\begin{bmatrix} \cos(\theta) & j\,Z_o \sin(\theta) \\ \dfrac{j}{Z_o}\sin(\theta) & \cos(\theta) \end{bmatrix}$ $\theta = \dfrac{\sqrt{\omega\,L\,\varepsilon_R}}{3\times10^{10}}$
Series L 	$\begin{bmatrix} 1 & j\,\omega\,L \\ 0 & 1 \end{bmatrix}$	Open Parallel Stub 	$\begin{bmatrix} 1 & 0 \\ \dfrac{j}{Z_o}\tan(\theta) & 1 \end{bmatrix}$ $\theta = \dfrac{\sqrt{\omega\,L\,\varepsilon_R}}{3\times10^{10}}$
Shunt L 	$\begin{bmatrix} 1 & 0 \\ \dfrac{-j}{\omega\,L} & 1 \end{bmatrix}$	Shorted Parallel Stub 	$\begin{bmatrix} 1 & 0 \\ \dfrac{-j}{Z_o \tan(\theta)} & 1 \end{bmatrix}$ $\theta = \dfrac{\sqrt{\omega\,L\,\varepsilon_R}}{3\times10^{10}}$
Series C 	$\begin{bmatrix} 1 & \dfrac{-j}{\omega\,C} \\ 0 & 1 \end{bmatrix}$	Open Series Stub 	$\begin{bmatrix} 1 & \dfrac{-j\,Z_o}{Z_o \tan(\theta)} \\ 0 & 1 \end{bmatrix}$ $\theta = \dfrac{\sqrt{\omega\,L\,\varepsilon_R}}{3\times10^{10}}$
Shunt C 	$\begin{bmatrix} 1 & 0 \\ j\omega\,C & 1 \end{bmatrix}$	Shorted Series Stub 	$\begin{bmatrix} 1 & \dfrac{j\,Z_o}{Z_o \tan(\theta)} \\ 0 & 1 \end{bmatrix}$ $\theta = \dfrac{\sqrt{\omega\,L\,\varepsilon_R}}{3\times10^{10}}$

Table II
Example Lumped Elements and Respective ABCD Representation

Gain Type	*Expression*	*Characteristics*	*Expression*
Transducer	$G_T = \frac{P_L}{P_{AVS}}$	Includes effects of both input and output terminations	$G_T = \frac{\lvert S_{2,1}\rvert^2 \left(1-\lvert\Gamma_S\rvert^2\right)\left(1-\lvert\Gamma_L\rvert^2\right)}{\lvert\left(1-S_{1,1}\Gamma_S\right)\left(1-S_{2,2}\Gamma_L\right)-S_{2,1}S_{1,2}\Gamma_S\Gamma_L\rvert^2}$
Operating	$G_P = \frac{P_L}{P_{IN}}$ $= G_T \Big\rvert_{\substack{MATCHED\\INPUT}}$	Is affected only by changing conditions at the output port	$G_P = \frac{\lvert S_{2,1}\rvert^2 \left(1-\lvert\Gamma_L\rvert^2\right)}{\left(1-\lvert\Gamma_{IN}\rvert^2\right)\lvert 1-S_{2,2}\Gamma_L\rvert^2}$ $where\ \Gamma_{IN} = S_{1,1} + \frac{S_{1,2}\, S_{2,1}\, \Gamma_L}{1-S_{2,2}\Gamma_L}$
Available	$G_A = \frac{P_{AVO}}{P_{AVS}}$ $= G_T \Big\rvert_{\substack{MATCHED\\OUTPUT}}$	Is affected only by changing conditions at the input port	$G_A = \frac{\lvert S_{2,1}\rvert^2 \left(1-\lvert\Gamma_S\rvert^2\right)}{\left(1-\lvert\Gamma_{OUT}\rvert^2\right)\lvert 1-S_{1,1}\Gamma_S\rvert^2}$ $where\ \Gamma_{OUT} = S_{2,2} + \frac{S_{1,2}\, S_{2,1}\, \Gamma_S}{1-S_{1,1}\Gamma_S}$
Insertion	$G_I = \frac{P_L}{P_{DL}}$	P_{DL} is power delivered to load with no matching; least useful of the four types	$G_I = \frac{\lvert S_{2,1}\rvert^2 \lvert 1-\Gamma_L\, \Gamma_S\rvert^2}{\lvert 1-\left(S_{1,1}\Gamma_S + S_{2,2}\, \Gamma_L + S_{1,2}\, S_{2,1}\, \Gamma_L\, \Gamma_S\right)+S_{1,1}S_{2,2}\Gamma_L\Gamma_S\rvert^2}$

Table III
Common Definitions of Power Gain

Questions to be Answered:

1. Significance of having noise sources both on the input side of the DUT versus having on at the input and one at the output?

2. How are the A and N matrices related? What is the significance of these definitions?

3. How/why are noise parameters related as they are to the A or N matrix?

4. Explain the basis for the equation: $A = k\, T_{phys} \left[identity(2) - S\, S^{*T} \right]$

5.

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